On the Matrix Equations $AH+HA^*=A^*H+HA=I$†

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ABSTRACT

This paper considers the simultaneous solution of the matrix equations $AH+HA^*=A^*H+HA=I$, where H is Hermitian. A full characterization of A in terms of a given H is obtained. Various results are also obtained for those H satisfying the equations with a given A.

I. INTRODUCTION

We consider the problem of finding those matrices A and Hermitian matrices H which simultaneously satisfy

$$AH+HA^*=I$$

and (1)

$$A^*H+HA=I.$$

This problem was suggested by Taussky [4]. Davis [3] obtained some results for normal or real matrices A. More general results were also obtained by Barker [1,2]. In this work we obtain additional results which include some results of Davis and Barker as special cases. In one instance, a result of Barker is proved under a less stringent hypothesis.

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We shall find it convenient to consider

$$(A+A^*)H+H(A+A^*)=2I \tag{2}$$

and

$$(A-A^*)H=H(A-A^*), \tag{3}$$

which are equivalent to (1). Then from either of (1) and (2) it is well known (e.g., Taussky and Wielandt [5]) that A, H and $A+A^*$ must be nonsingular. Also (3) obviously states that H and the skew-Hermitian part of A must commute if (1) is to hold.

II. CHARACTERIZATION OF A FOR GIVEN H

Suppose that U is a unitary matrix which diagonalizes H, i.e.,

$$U^*HU=G=\operatorname{diag}(\gamma_1,\ldots,\gamma_n), \tag{4}$$

where γ_j are the eigenvalues of H. Then define

$$B=U^*(A+A^*)U$$

and (5)

$$S=U^*(A-A^*)U,$$

so that (2) and (3) become

$$BG+GB=2I \tag{6}$$

and

$$SG=GS. \tag{7}$$

For a given matrix H we may completely characterize the matrix A by

THEOREM 1. *Let γ_j, $j=1,\ldots,n$, be the eigenvalues and U a unitary diagonalizer of a given Hermitian matrix H. Then the Hermitian part of a* *matrix A which satisfies* (1) *is given by*

$$A + A^* = H^{-1} + UB_1U^*, \tag{8}$$

where B_1 is a Hermitian matrix whose entries are zero except when $\gamma_i + \gamma_j = 0$, in which case the (i,j) entry of B_1 is arbitrary. Furthermore, the skew-Hermitian part of A is given by

$$A - A^* = US_1U^*, \tag{9}$$

where S_1 is a skew-Hermitian matrix whose diagonal entries are arbitrary imaginary numbers and whose (i,j)th entry is zero whenever $\gamma_i \neq \gamma_j$ and is arbitrary otherwise.

Proof. If b_{ij} is an element of the Hermitian matrix B defined by (5), then (6) yields

$$b_{ij}(\gamma_i + \gamma_j) = 2\delta_{ij}, \qquad i,j = 1,\dots,n.$$

Since H is nonsingular, $b_{ii} = 1/\gamma_i$, $b_{ij} = 0$ if $\gamma_i + \gamma_j \neq 0$, and $b_{ij} = b_{ji}$ is arbitrary whenever $\gamma_i + \gamma_j = 0$. Then $B = G^{-1} + B_1$, which using (4) and (5) results in (8). If s_{ij} is an element of the skew-Hermitian matrix S, then (7) can be expressed as

$$s_{kl}(\gamma_k - \gamma_l) = 0, \qquad k,l = 1,\dots,n,$$

so that s_{kk} is arbitrary, $s_{kl} = 0$ whenever $\gamma_k \neq \gamma_l$, and s_{kl} is arbitrary whenever $\gamma_k = \gamma_l$. Thus, with $S_1 = S$, (5) results in (9). ■

A consequence of Theorem 1 is

COROLLARY 1.1. *If $\gamma_i + \gamma_j \neq 0$ for all i and j and if* (1) *holds, then A and H commute and A is normal.*

Proof. If $\gamma_i + \gamma_j \neq 0$ for all i and j, then $B_1 = 0$ and $A + A^* = H^{-1}$, so that $A + A^*$ trivially commutes with H. But from (3), $A - A^*$ also commutes with H, so that A and H commute. Furthermore, if $A - A^*$ commutes with H, it also commutes with H^{-1}, so that $A - A^*$ commutes with $A + A^*$, and therefore A is normal. ■

This corollary was proven by Barker [1] with the additional (and unnecessary) assumption of $\gamma_i \neq \gamma_j$. Furthermore, the proof of the corollary contains the proof of the following theorem, which is also contained in Barker [1]:

THEOREM 2. *If* (1) *holds for some Hermitian matrix* H *and if* $(A+A^*)$ *and* H *commute, then* A *is normal.*

We also have:

COROLLARY 2.1. *If* (1) *holds for some Hermitian matrix* H *and if* A *and* H *commute, then* A *is normal.*

Proof. If A and H commute, then so do $A+A^*$ and H, so that by Theorem 2, A is normal. ∎

III. CHARACTERIZATION OF H FOR GIVEN A

We now seek Hermitian matrices H such that (1) is satisfied for a given complex matrix A. We first characterize such H using the diagonalizer of the skew-Hermitian part of the given matrix A:

THEOREM 3. *Let* V *be a unitary diagonalizer of the skew-Hermitian part of a given matrix* A. *Let* $B=V^*(A+A^*)V$. *Then if the* (i,j) *element* b_{ij} *of* B *is zero whenever* $b_{ii}+b_{jj}\neq 0$, *then* H *given by*

$$H=V\operatorname{diag}\left(b_{11}^{-1},\dots,b_{nn}^{-1}\right)V^* \tag{10}$$

is a solution of (1).

Proof. With H given by (10), (2) can be expressed as

$$b_{ij}\left(\frac{1}{b_{ii}}+\frac{1}{b_{jj}}\right)=2\delta_{ij},\qquad i,j=1,\dots,n, \tag{11}$$

which is an identity for $i=j$. For $i\neq j$, (11) becomes

$$b_{ij}\left(b_{ii}+b_{jj}\right)=0,$$

which is again an identity due to the hypothesis of the theorem. Furthermore, (3) holds trivially, since under the unitary transformation V, H and $A - A^*$ are both diagonalized. Since (2) and (3) hold, so does (1). ■

REMARK. If the eigenvalues of $A - A^*$ are distinct, then the hypothesis about the elements of B is necessary for a solution H of (1) to exist. Furthermore, if such a solution does exist, it is uniquely given by (10).

We now consider the case where $A + A^*$ and $-(A + A^*)$ have no common eigenvalue:

THEOREM 4. *Let β_i, $i = 1, \dots, n$, be the eigenvalues and W be a unitary diagonalizer of the Hermitian part of a given matrix A. Let $S = W^*(A - A^*)W$. Then if $\beta_i + \beta_j \neq 0$ for all i and j, (1) has a solution H only if $s_{ij} = 0$ whenever $\beta_i \neq \beta_j$. If this is the case, then $H = (A + A^*)^{-1}$.*

Proof. Let

$$G = W^* H W. \tag{12}$$

Then (2) can be expressed as

$$g_{ij}(\beta_i + \beta_j) = 2\delta_{ij}, \qquad i, j = 1, \dots, n,$$

where g_{ij} is an element of G, so that

$$g_{ii} = 1/\beta_i \quad \text{and} \quad g_{ij} = 0, \quad i \neq j. \tag{13}$$

Then (3) can be written as

$$s_{ij}(g_{ii} - g_{jj}) = 0, \qquad i, j = 1, \dots, n,$$

or from (13),

$$s_{ij}(\beta_i - \beta_j) = 0, \qquad i, j = 1, \dots, n, \tag{14}$$

where s_{ij} is an element of S. Therefore either $s_{ij} = 0$ or $\beta_i = \beta_j$ for a solution to exist. Then, if (14) is valid for all i and j, (12) and (13) yield $H = (A + A^*)^{-1}$. ■

Another result of Barker [1] follows immediately from this theorem:

COROLLARY 4.1. *If* (1) *holds for some Hermitian matrix* H *and if* $A + A^*$ *and* $-(A + A^*)$ *have no common eigenvalue, then* A *is normal.*

Proof. By Theorem 4, if $A + A^*$ and $-(A + A^*)$ have no common eigenvalue and (1) holds, then $H = (A + A^*)^{-1}$, so that trivially $A + A^*$ and H commute. Then, by Theorem 2, A is normal. ■

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